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SHADOW LINKS AND FACE MODELS OF STATISTICAL MECHANICS

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Abstract

We introduce a new geometric technique enabling one to present links in oriented 3-manifolds, which are S^1-fibrations over oriented surfaces, by configurations of loops on the surfaces equipped with some additional data. This technique naturally leads to a purely 2-dimensional notion of shadow links on surfaces. We use IRF-models, based on the quantum $6j$-symbols associated with the Lie algebra $\mathrm{Sl}_2(\mathbb{C})$, to construct invariants of shadow links generalizing the Jones polynomial of links in the 3-sphere S^3.

Introduction

Since the appearance of the Jones polynomial for links in the 3-sphere S^3, considerable efforts have been spent to construct analogous invariants for links in other 3-manifolds. An important breakthrough was made by E. Witten [16] who defined (on the physical level of rigour) Jones-type invariants of links in arbitrary closed 3-manifolds. Witten's approach is based on quantum field theory with a nonabelian Chern-Simons action.

A mathematical construction of "quantum" invariants of links generalizing the Jones polynomial was given in [10]. This construction is based on the representation theory of quantum groups and the surgery theory of manifolds. The surgery theory, developed in dimension 3 by V. Rochlin, W. Lickorish, R. Kirby, R. Fenn, and C. Rourke, enables one to reduce the study of links in closed 3-manifolds to the case of links in S^3 where the technique of R-matrices and categories of tangles is applicable (see [2], [9]–[14]). The invariants of links in 3-manifolds obtained in this way satisfy the same formal properties as the Witten invariants and therefore may be viewed as a mathematical realization of Witten's program.

Another approach to quantum invariants of compact 3-manifolds and links in these manifolds has been presented in [15]. The construction in [15] is based on the theory of quantum $6j$-symbols developed in [6] and

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the theory of special spines of 3-manifolds. This approach allows us to derive "quantum" invariants of 3-manifolds directly from triangulations.

In this paper we present yet another approach which enables us to define and study quantum invariants of framed links which lie in the total spaces of oriented S^1-bundles over oriented surfaces. With this view, we introduce a new geometric technique of link shadows. Essentially, a shadow on a surface F is a finite collection of loops on F lying in general position, each component of the complement of the loops being equipped with a number. Links in oriented circle bundles over an oriented closed surface F are shown to produce shadows on F. When $F = S^2$ one may reconstruct each link (up to isotopy) from its shadow. The technique of shadows is applicable, in particular, to classical links in S^3 since the sphere S^3 fibers over S^2 via the Hopf mapping.

We develop an abstract theory of shadows on a surface, including the notions of isotopy and regular isotopy of shadows. Isotopy classes of shadows are called shadow links or, briefly, shlinks. Regular isotopy classes of shadows are called framed shlinks. The whole approach runs parallel to the well-known exposition of knot theory in terms of link diagrams and Reidemeister moves (see [8]).

We use slightly modified IRF-models (interaction round a face models) of statistical mechanics to construct $\mathbb{C}$-valued invariants of framed shlinks. The model in question is based on the so-called $(q-6j)$-symbols, where q is a complex root of 1 (see [6]). Combining this model with the technique of link shadows, we get Jones-type invariants of framed links in oriented S^1-bundles over oriented surfaces. In the case of links in S^3 we reconstruct in this way the values of the Jones polynomial in the corresponding roots of 1. We also develop a more general theory which produces $\mathbb{C}$-valued invariants of colored framed shlinks.

From the viewpoint of 3-dimensional topology, the constructions of the present paper have a somewhat limited area of applications. Indeed, the majority of closed 3-manifolds do not fiber over surfaces. On the other hand the present approach suggests quite a new view of the quantum invariants of links. The appearance of IRF-models is also remarkable.

Our constructions shed some additional light on the original Jones polynomial of links in S^3. Indeed, as it turns out, this polynomial (or at least its values in the complex roots of 1) may be extended to the set of real (and even complex) framed shlinks on S^2, which is a kind of completion of the set of isotopy types of framed links in S^3. This extension might be of use in the search of nontrivial knots in S^3 with Jones polynomial 1. It would be of interest to find a non-trivial shadow knot with this property.

At the moment of writing, the relationships between the approaches of [10], [15], and the present approach are not clear, though in principle they must be related.

The standard method of presenting links in S^3 by plane link diagrams appeals to the projection $\mathbb{R}^3 \to \mathbb{R}^2$. It was the idea to try the Hopf mapping $S^3 \to S^2$ instead of this projection which gave the initial impetus to this work. The application of IRF-models to shadows was inspired by the results of [6]. (It should be noted that our notion of shadow link differs from the notion of link in the shadow world used by Kirillov and Reshetikhin [6]: their links are presented by ordinary diagrams with overcrossings and undercrossings, without any hints to gleams of regions.)

For background information on the invariants of links in S^3 related to statistical mechanical models, the reader is referred to [4] and [7].

The paper is organized as follows. In §1 we define shadows and shadow links, and discuss their simple properties. In §2 we apply the classical technique of link diagrams to show that each link in the cylinder $F \times \mathbb{R}$ over any oriented surface F canonically gives rise to a shlink on F. We also formulate Theorem 2.1 which establishes a correspondence between links in S^3 an shlinks on S^2. In §3 we establish a more general correspondence between links in S^1 bundles over surfaces and integral shlinks. In §4 we present a version of the results of §§1–3 for framed links and framed shlinks. In §5 we introduce the relevant IRF-models and the state model invariants of colored (framed) shlinks. In §6 we consider the IRF-model based on $(q - 6j)$-symbols and the corresponding invariants of colored shlinks. In §7 a brief description of the so-called shadow 3-manifolds is presented.

Notation. Throughout the paper the symbol F denotes an oriented surface (possibly with boundary, noncompact, etc.).

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1. Shadows and shlinks

Throughout this section the symbol A denotes a fixed abelian group containing the group of integers: $A \supset \mathbb{Z}$. (Important example: $A = \mathbb{Z}$.)

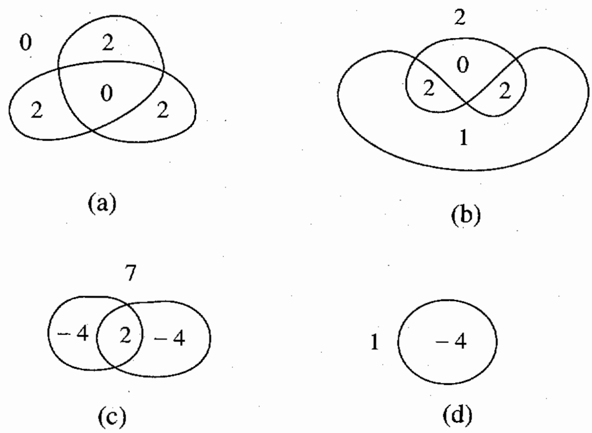

FIGURE 1

a. Shadows. A shadow over A (or, briefly, an A-shadow) on the surface F is a finite family of immersed closed curves in F with only double transversal crossings (and self-crossings) such that each connected component of the complement of these curves in F is equipped with an element of A. These connected components are called regions of the shadow. The associated elements of A are called gleams of the regions. Examples of shadows on the plane $\mathbb{R}^2$ are given in Figure 1.

Sometimes it is convenient to assume that the gleam of a region may be concentrated in certain parts of the region. We will use the following convention: when several elements of A are attached to (certain parts of) a region, then the gleam of this region equals the sum of these elements.

The total gleam of a shadow is defined to be the sum of the gleams of all its regions minus twice the number of crossing points. (To justify this definition one may think that each crossing point has the negative gleam -2). For example, the gleams of shadows presented in Figure 1 are equal respectively to 0, 1, -3, -3.

The crossing points of a shadow s dissect the underlying loops of s into imbedded segments. These segments will be called edges of s.

b. Isotopies of shadows. Shlinks. We define three local moves on shadows $S1$, $S2$, $S3$ (see Figure 2). The symbols $x, y, z, \cdots$ in Figure 2 represent the gleams of the corresponding regions (under the convention exhibited in subsection a). These $x, y, z, \cdots$ may take arbitrary values in the group A. The integers $0, 1, 2 \in \mathbb{Z} \subset A$ clearly play a distinguished role in the moves $S1$, $S2$, $S3$. A knot-theoretic explanation of these moves will be given in §2.

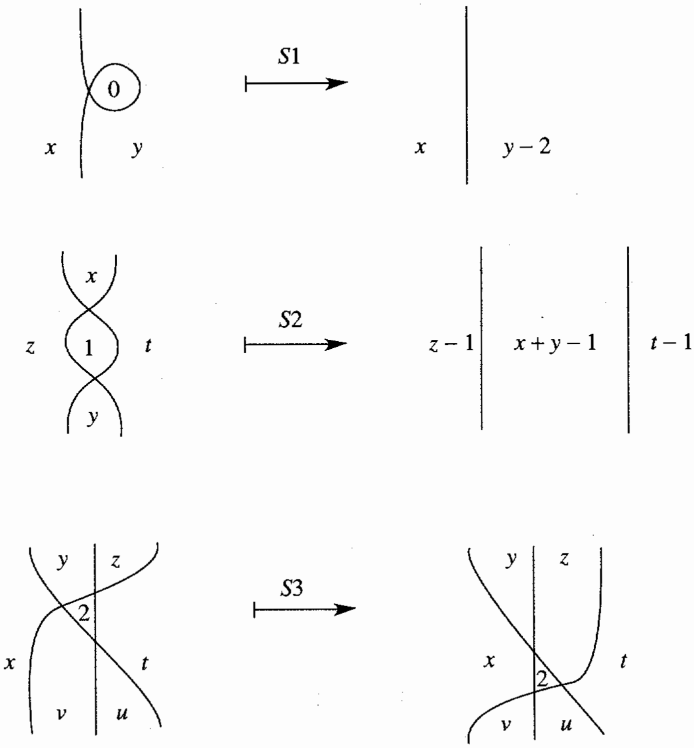

FIGURE 2

The moves $S1$ and $S3$ have the obvious inverses, denoted by $(S1)^{-1}$ and $(S3)^{-1}$. The move $S2$ has many inverses since there are many pairs x, y with the given sum. Note that this nonuniqueness is not essential whenever x, y are attached to the same region of the shadow. By the move $(S2)^{-1}$ we will mean an arbitrary inverse to $S2$.

Two shadows on the surface F are called isotopic if they can be transformed into each other by a sequence of moves $S1$, $S2$, $S3$ and their inverses. An example of isotopy is given in Figure 3 (next page).

Isotopy classes of shadows on F will be called shadow links or, briefly, shlinks on F (over the group A).

Remark that the moves $S1$, $S2$, $S3$ preserve the total gleam. Thus the total gleam is an isotopy invariant of shadows, and we may speak of the (total) gleam of a shlink.

The shadows and shlinks over the additive group of integers $A = \mathbb{Z}$ are called integral ones. The shadows and shlinks over the additive group of

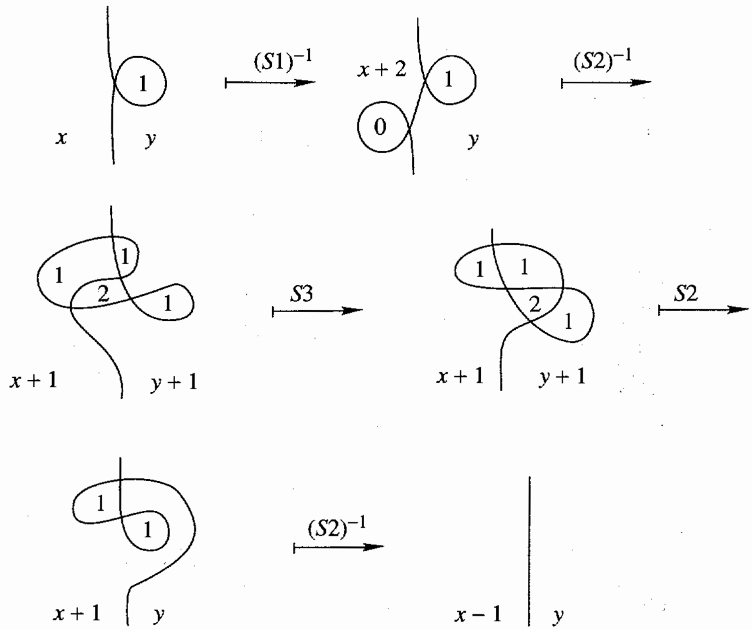

FIGURE 3

rational numbers (resp. real numbers, complex numbers, quaternions) are called rational (resp. real, complex, quaternionic) ones.

c. Transition of shlinks. Let $f: F' \hookrightarrow F$ be an orientation-preserving imbedding of a surface F' into the surface F such that the set $F \backslash f(F')$ is nonempty and connected. Fix an element a of the group A. We may transfer each shadow on F' into a shadow on F via f assuming that the set $F \backslash f(F')$ contributes a to the gleam of the region which contains it. Clearly, isotopic shadows on F' are transferred to isotopic shadows on F. Thus, each shlink K on F' extends to a shlink on F, denoted by $f_a(K)$.

In particular, take f to an imbedding $\mathbb{R}^2 \hookrightarrow \operatorname{Int} F$. Then each shlink K on $\mathbb{R}^2$ with a gleam k extends to a shlink $K_a = f_a(K)$ on F with the gleam $k + a$. If F is connected, then all imbeddings $\mathbb{R}^2 \hookrightarrow \operatorname{Int} F$ are isotopic and the shlink K_a depends only on K and a.

d. Subshadows and subshlinks. If a shadow s on F is formed by n loops $\gamma_1, \cdots, \gamma_n$, then for each subset $E \subset \{1, 2, \cdots, n\}$ we define a subshadow s_E of s. It is formed by the curves $\{\gamma_e\}_{e\in E}$. The gleam of a region X of s_E is defined to be $a(X) - b(X) - 2c(X)$, where $a(X)$ is the sum of gleams of regions of s contained in X, $b(X)$ is the number of crossing points of s lying on the boundary $\partial\overline{X}$ of the closure $\overline{X}$ of X and distinct from the vertices of $\partial\overline{X}$, and $c(X)$ is the number of crossing points of s lying in the interior of $\overline{X}$. Note that the total gleam of s_E is equal to that of s.

In particular, when E is a 1-element set $\{e\}$ we get a shadow s_e formed by the loop γ_e and called a component of s.

It is easy to check that each isotopy relating two shadows induces an isotopy of their corresponding subshadows. Therefore we may speak of subshlinks of a given shlink. The one-component subshlinks of a shlink are called its components. For example the shlink on S^2 shown in Figure 1(c) has two isotopic components, shown in Figure 1(d).

e. Remarks. 1. The notion of shlink is purely 2-dimensional. Generally speaking, there is no natural 3-manifold in which a given shlink may sit. The only global datum related to a shlink, which is preserved under passage to subshlinks, is the pair (the surface F, the total gleam). When the total gleam is not an integer there seems to be no reasonable way to associate a 3-manifold with such a pair. On the other hand the integral shlinks are intricately related to links in certain 3-manifolds (see §§2, 3).

2. If the group A contains the group $\frac{1}{2}\mathbb{Z}$ of integers and half-integers, then we may distribute the gleam -2 of each crossing point to the four incident regions (see Figure 4) and forget about the gleams of crossing points. This gives an equivalent but slightly different language to describe shadows and shlinks. This language is often more convenient. For example, the total gleam of a shadow is just the sum of the modified gleams of regions. Also the modified gleam of a region X of a subshadow of a shadow s is just the sum of the modified gleams of regions of s contained in X. However, this approach does not cover the case of integral shlinks.

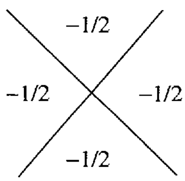

FIGURE 4

3. If the ground abelian group A is imbedded into another abelian group A' then each shlink over A determines in the obvious way a shlink over A'. It is of interest to study injectivity and surjectivity properties of this mapping from the set of A-shlinks into the set of A'-shlinks. Denote this mapping by $\varphi(A, A')$. There is a simple criterion for injectivity of this mapping: If there exists an additive homomorphism $r: A' \to A$ with $r|A = \mathrm{id}_A$ then $\varphi(A, A')$ is injective. Indeed, if two shadows over A are isotopic in the class of A'-shadows then we may apply r to the gleams of regions of all intermediate shadows and get in this way an isotopy over A. Since each isotopy involves only a finite number of shadows and their regions, it suffices to meet the following weaker condition: for any finite subset $G \subset A'$ there exist a subgroup A'' of A and a homomorphism $r: A'' \to A$ such that $A'' \supset A \cup G$ and $r|A = \mathrm{id}_A$. These observations imply that in the case when A is the additive group of a certain field and A' is a module over this field then $\varphi(A, A')$ is injective. For instance, the sets of rational, real, complex, and quaternionic shlinks are injectively imbedded each in the next one.

It is not known to the author whether the mapping $\varphi(\mathbb{Z}, Q)$ is injective.

4. Each complex shadow s has the real and imaginary parts $\mathrm{Re}\, s$ and $\mathrm{Im}\, s$ obtained by taking real and imaginary parts, respectively, of the gleams of regions. Both $\mathrm{Re}\, s$ and $\mathrm{Im}\, s$ are real shadows. The real parts of isotopic complex shadows are clearly isotopic over $\mathbb{R}$. In other words, each complex shlink has a real part which is a real shlink. Similarly, each quaternionic shlink has a complex part which is a complex shlink.

2. Links in cylinders and their shadows

a. Shadows of links in $F\times\mathbb{R}$. We show in this subsection that each link K lying in the cylinder $F\times\mathbb{R}$ canonically produces an integral shlink $S(K)$ on F. The construction of $S(K)$ goes via link diagrams and explains the origins of the moves $S1$, $S2$, $S3$.

By a link in a 3-manifold N we mean a finite collection of mutually disjoint circles smoothly imbedded in $N \backslash \partial N$. Two links are called isotopic if they may be smoothly deformed into each other in the class of links in N.

Links in $F \times \mathbb{R}$ may be presented by link diagrams on F in the same fashion in which links in $\mathbb{R}^3$ may be presented by plane link diagrams. A link diagram on F is a finite family of immersed closed curves in F with only double transversal crossings equipped with an additional structure: at

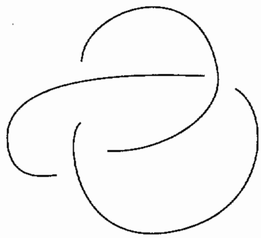

FIGURE 5

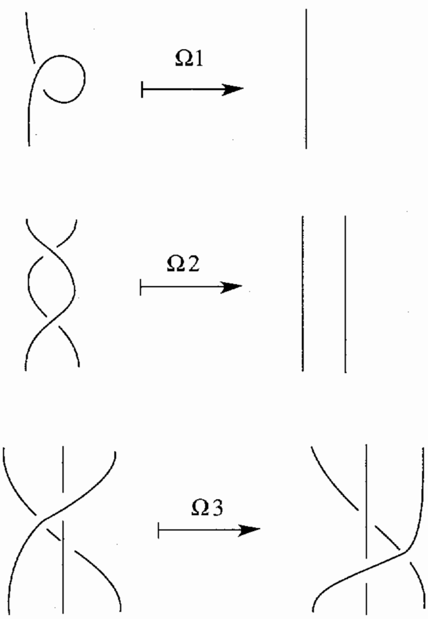

FIGURE 6

each crossing one branch is cut out and considered as the lower one (the undercrossing), the second branch being considered as the upper one (the overcrossing). An example of a link diagram on $\mathbb{R}^2$ is given in Figure 5. This diagram presents a knot in $\mathbb{R}^3$ (the trefoil).

An immediate generalization of a Reidemeister theorem [8] states that two link diagrams on F present isotopic links if and only if they can be obtained from each other by a finite sequence of Reidemeister moves $\Omega 1$, $\Omega 2$, $\Omega 3$ (see Figure 6) and their inverses.

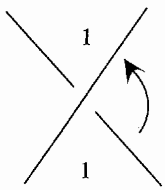

FIGURE 7

Each link diagram $\mathscr{D}$ on F canonically produces a shadow $s(\mathscr{D})$ on F. It has the same underlying family of closed curves as $\mathscr{D}$. To define gleams of regions we put $+1$ in two opposite regions incident to each crossing point as in Figure 7. The choice of these two regions is determined both by the diagram $\mathscr{D}$ and the fixed orientation in F, also shown in Figure 7. The gleam of a region is defined to be the number (or the sum) of $+1$'s located inside this region. The total gleam of $s(\mathscr{D})$ clearly equals zero.

For example, the diagram shown in Figure 5 gives rise to the shadow pictured in Figure 1(a).

It is straightforward to see that the passage to shadows transforms the Reidemeister moves $\Omega 1$, $\Omega 2$, $\Omega 3$ respectively into $S1$, $S2$, $S3$. Therefore if two link diagrams on F present isotopic links, then the corresponding shadows are isotopic. This shows that each link K in $F \times \mathbb{R}$ gives rise to an integral shlink on F with total gleam 0. This shlink will be denoted by $S(K)$.

Generally speaking one cannot reconstruct a link in $F \times \mathbb{R}$ from the corresponding shlink on F. For example, for any two link diagrams $\mathscr{D}$, $\mathscr{D}'$ on F one may define their "product" $\mathscr{D}\mathscr{D}'$ by positioning $\mathscr{D}$ over $\mathscr{D}'$. It is easy to see directly (or to deduce from the results of §3) that the diagrams $\mathscr{D}\mathscr{D}'$ and $\mathscr{D}'\mathscr{D}$ produce the same shadow on F. However, the links in $F \times \mathbb{R}$ represented by $\mathscr{D}\mathscr{D}'$ and $\mathscr{D}'\mathscr{D}$ may be nonisotopic.

In the case $F = \mathbb{R}^2$ the diagrams $\mathscr{D}\mathscr{D}'$ and $\mathscr{D}\mathscr{D}'$ represent isotopic links and, indeed, it follows from Corollary 2.3, formulated below, that link diagrams on $\mathbb{R}^2$ with the same shadow always present isotopic links.

The product construction is, possibly, the only source of nonisotopic links in $F \times \mathbb{R}$ which induce the same shlink on F. In particular, I do not know if there exist nonisotopic knots K_1, $K_2 \subset F \times \mathbb{R}$ with $S(K_1) = S(K_2)$.

b. Shadows of links in $\mathbb{R}^3$. Recall first that the inclusion $\mathbb{R}^3 \hookrightarrow S^3$ induces a bijective correspondence between isotopy types of links in $\mathbb{R}^3$

and isotopy types of links in S^3. Thus the topological theories of links in $\mathbb{R}^3$ and in S^3 are equivalent.

According to the results of subsection a, each link $K \subset \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ gives rise to an integral shlink $S(K)$ on $\mathbb{R}^2$ with total gleam 0.

For any integer n the shlink $S(K)$ extends to an integral shlink $S(K)_n$ on $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Recall that $S(K)_n$ is the isotopy type of the shadow on S^2 obtained from the shadow $s(\mathscr{D})$ of any diagram $\mathscr{D}$ of K by adding n to the gleam of the region which contains the point $\{\infty\}$.

Theorem 2.1. *For any integer n the formula $K \mapsto S(K)_n$ defines an injective mapping of the set of isotopy types of links in $\mathbb{R}^3$ in the set of integral shlinks on S^2 with total gleam n. This mapping is surjective if and only if $n = \pm 1$.*

This theorem will be proven in §3.d.

Corollary 2.2. *The formula $K \mapsto S(K)_1$ establishes a bijective correspondence between the set of isotopy types of links in $\mathbb{R}^3$ and the set of integral shlinks on S^2 with total gleam 1.*

This corollary may be viewed as an alternative description of the set of links in $\mathbb{R}^3$ or S^3 in terms of integral shadows on S^2 and their moves $S1$, $S2$, $S3$. For example, the shadow on S^2 depicted in Figure 1(b) represents via the correspondence of Corollary 2.2 the same trefoil as the diagram in Figure 5.

This approach has both advantages and disadvantages. The main disadvantage is that when looking at a shadow on S^2 it is rather hard to visualize the corresponding link in $\mathbb{R}^3$. The important merit of the approach is that having a finite collection of generic loops on S^2 one may construct an infinite number of links in $\mathbb{R}^3$ just by varying the gleams of the regions in $\mathbb{Z}$ (keeping the total gleam 1). Moreover, varying the gleams of regions in $\mathbb{R}$ one gets continuous families of real shlinks which "connect" the genuine links in $\mathbb{R}^3$.

Corollary 2.3. *The formula $K \mapsto S(K)$ defines an injective mapping of the set of isotopy types of links in $\mathbb{R}^3$ into the set of integral shlinks on $\mathbb{R}^2$ with total gleam 0.*

c. Remarks. 1. Putting orientations on the underlying loops of shadows, one gets the notion of oriented shadows and oriented shlinks. The results of §§1, 2 as well as the results of §§3, 4 may be straightforwardly transferred to this oriented setting.

2. Corollary 2.2 implies that all standard invariants of links (the Alexander polynomial, signatures, etc.) should be computable from the corresponding integral shlinks on S^2. It would be most interesting to get

explicit formulas for these invariants since such formulas may suggest extension of the invariants to real or even complex shlinks.

3. Links in S^1-bundles and their shadows

a. Shadows of links. Let F be an oriented closed connected surface and let $p: N \to F$ be an oriented circle bundle over F. A link $K \subset N$ with components $K_1, \cdots, K_m$ is called generic (with respect to p) if K is transversal to the fibers of p, and the immersed curves $p(K_1), \cdots, p(K_m) \subset F$ have only double transversal crossings. We will associate with such a generic link K a certain integral shadow on F whose underlying family of curves is the family $\{p(K_1), \cdots, p(K_m)\}$. This shadow will be denoted by $s(K)$ and called the shadow of K.

We need some preliminary constructions. We assume that the circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

freely acts on N so that the orbits of the action coincide with the fibers of p, and the orientation of each fiber is induced by the counterclockwise orientation of S^1. In particular, $-1 \in S^1$ acts as a free involution on N transforming each point $x \in N$ into the "opposite"point $(-1)x$. Let N' be the quotient 3-manifold $N/(-1)$. We have the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{q} & N' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & F & \end{array}$$

where q is the projection $N \to N/(-1)$, and p' is the oriented circle fiber bundle over F induced by p. The projection q maps each fiber of p onto a fiber of p' as a two-sheeted covering of degree $+2$.

Let us call an isotopy of a link in N vertical if during this isotopy each point of the link moves along its S^1-orbit. In particular, vertically isotopic links have the same projection in F.

It is clear that each generic link $K \subset N$ may be vertically isotoped to a generic link L such that the following condition $(*)$ holds:

$(*)$ any two distinct points of L which lie over the same crossing point of $p(L) = p(K)$ are opposite to each other.

The set $q(L)$ is obtained from L by identification of the opposite points which are exactly the points projecting to the same point in F. Therefore the mapping p' projects the graph $q(L)$ bijectively onto $p(L) = p(K)$. Denote the inverse bijection

$$p(K) \to q(L) \subset N'$$

by r. Clearly r is a section of p' over $p(K)$.

Cut out F along $p(K)$. Let X be one of the resulting pieces. Clearly, X is a connected 2-manifold with $\partial X \neq \emptyset$, and $\operatorname{Int} X$ is a connected component of $F \backslash p(K)$. The orientation of F induces an orientation in X. The bundle p reduces to an oriented circle bundle over X denoted by p'_X. The section r of p' induces a section of p'_X over ∂X. Denote this latter section by $\tilde{r}$. There is an obvious obstruction to extending $\tilde{r}$ over X. Namely, the bundle p'_X is actually trivial so that we may identify it with the projection $S^1 \times X \to X$. Composing the section $\tilde{r}: \partial X \to S^1 \times \partial X$ with the projection $S^1 \times X \to S^1$ we get a mapping $\partial X \to S^1$, whose degree, computed with respect to the orientation of ∂X induced by that of X, is the obstruction mentioned above. Denote this degree by α_X. The integer α_X does not depend on the choice of trivialization of p'_X: two such choices differ by an element of $H^1(X)$, and all these elements annihilate the fundamental class $[\partial X] = \partial[X, \partial X]$. The number α_X also does not depend on the choice of the link L, since any two such links may be related by a vertical isotopy in the class of generic links satisfying $(*)$.

Let β_X be the number of corners of the region X. (A corner of a region is shaded in Figure 8.) In other words, β_X is the number of crossing points of $p(K)$ incident to X (possibly with multiplicity). It is easy to see that the integer $\beta_X - \alpha_X$ is always even. We define the gleam of X to be $(\beta_X - \alpha_X)/2$. This gives us the integral shadow $s(K)$. The sum of the numbers $\beta_X/2$ over all regions X equals twice the number of crossing points of $p(K)$, and therefore the total gleam of $S(K)$ equals the sum of the numbers $-\alpha_X/2$ over all regions X. The latter sum may be easily identified with $-\chi(p')/2 = -\chi(p)$, where $\chi(\xi)$ is the Euler number of the 2-dimensional real vector bundle over F associated with the oriented circle bundle ξ.

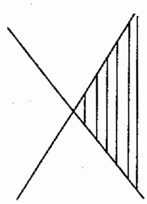

FIGURE 8

Lemma 3.1. *If two generic links in N are isotopic, then their shadows are also isotopic.*

This lemma enables us to associate with every link $K \subset N$ a certain shlink $S(K)$ on F. Namely, take an arbitrary generic link K' in F isotopic to K, and define $S(K)$ to be the shlink presented by the shadow $s(K')$. Lemma 3.1 ensures that $S(K)$ does not depend on the choice of K'. The total gleam of $S(K)$ is equal to $-\chi(p)$.

The constructions of the present section applied to $N = F \times S^1$ generalize the constructions of §2.a. Indeed, let us imbed $F \times [0, 1]$ into $F \times S^1$ by the mapping

$$(f, t) \mapsto (f, e^{\pi\sqrt{-1}t}) : F \times [0, 1] \to F \times S^1.$$

Each link diagram $\mathscr{D}$ on F represents a link, say, $K_{\mathscr{D}} \subset F \times [0, 1]$ which is obviously a generic link in $F \times S^1$. It is easy to see that the shadows $s(\mathscr{D})$ and $s(K_{\mathscr{D}})$ coincide. Therefore, for any link $K \subset F \times [0, 1] \subset F \times S^1$ the definitions of the shlink $S(K)$ given in §2.a and in the present section are equivalent. Note that the Euler number of the projection $F \times S^1 \to F$ is equal to zero.

b. Proof of Lemma 3.1. Let B be a 2-disc and let $B \times [0, 1] \subset N$ be a cylinder lying in N. We will say that this cylinder is normal if the fibers of p either do not intersect this cylinder or intersect it in vertical segments $b \times [0, 1]$ with $b \in B$. For any normal cylinder $B \times [0, 1] \subset N$ the projection p maps $B = B \times 0$ homeomorphically onto a disc in F whose preimage $p^{-1}(p(B))$ is a solid torus containing $B \times 0$ as a meridianal disc.

An arbitrary isotopy of a link in N may be presented as a composition of small isotopies which proceed entirely inside normal cylinders in N. This fact easily implies that any two isotopic generic links in N may be related by a sequence of (i) vertical isotopies, (ii) local isotopies, each of which proceeds inside a normal cylinder $B \times [0, 1] \subset N$ such that the link under isotopy never meets the cylinder

$$p^{-1}(p(B \times 0)) \backslash (B \times [0, 1]).$$

The vertical isotopies do not change the shadow of the generic link. The part of the link lying inside a normal cylinder $B \times [0, 1]$ with (possibly) some ends in $\partial B \times [0, 1]$ may always be represented by a tangle diagram on the disc B with the ends in ∂B. The local isotopies of type (ii) proceeding inside $B \times [0, 1]$ may be presented as compositions of Reidemeister moves on such tangle diagrams with fixed ends in ∂B. As in §2.a, when one applies the moves $\Omega 1 - \Omega 3$ to diagrams in B, the corresponding shadows on F are changed respectively by $S1-S3$. This implies the claim of the lemma.

c. Action of $H_1(F)$ on the set of links. If the genus of F is positive, then there is no chance to reconstruct a link in N from its shadow. For example, let $N = F \times S^1$ and let K be a knot in N, which projects to a simple closed curve γ in F with the connected complement $F \backslash \gamma$. The shadow of K is formed by the loop γ with the zero gleam of the only region $F \backslash \gamma$. Let the knot K in N be obtained from K by replacing a small segment α on K by the arc α' in $p(\alpha) \times S^1$ shown in Figure 9. Clearly, the homology classes of K and K' in $H_1(N; \mathbb{Z}/2\mathbb{Z})$ differ, and therefore K' is not isotopic to K. On the other hand, K' projects to the same loop γ an the gleam of the only region $F \backslash \gamma$ of $s(K')$ equals zero. Indeed, the glued in arc α' contributes twice to the gleam of $F \backslash \gamma$ with different signs.

To handle this phenomenon we introduce an action of the group $H_1(F) = H_1(F; \mathbb{Z})$ on the set of isotopy classes of links in N. (This construction works for an arbitrary oriented surface F, not necessarily closed.)

Let K be a generic link in N and let β be an oriented closed (possibly self-intersecting) curve on F, which presents a class $[\beta] \in H_1(F)$. Deforming β, if necessary, we may assume that β intersects $p(K)$ transversally in a finite number of points distinct from the crossing points of $p(K)$. Let $\alpha = [a, b]$ be a small segment of K such that $p(\alpha)$ contains exactly one intersection point c of $p(K)$ and β. Assume that $p(a)$ lies to the left of β and $p(b)$ lies to the right of β. Replace α by the arc α' shown in Figure 9. We will call this transformation of K a fiber-fusion over the "blowing up" point c. (Note that to apply a fiber-fusion to K we must choose a noncrossing point of $p(K)$ to be blowed up and an orientation of $p(K)$ at this point.) Applying fiber-fusion to K over all points of

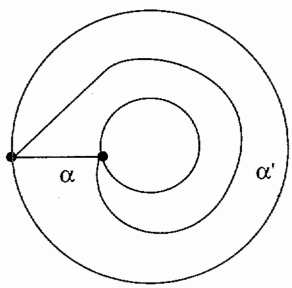

FIGURE 9

$p(K) \cap \beta$ we get a new generic link K' with $p(K') = p(K)$. Note that the shadows of K and K' coincide. Indeed, each time when β enters a region X of $s(K)$ it must wander inside the region and finally leave it so that the contributions of the newly inserted arcs to the gleam of X cancel with each other. Thus, $s(K') = s(K)$.

Let us show that the isotopy type of the link K' depends only on the isotopy type of K and the homology class of β. Note first that the isotopy type of K' is not changed when we deform β through a crossing of $p(K)$. Also, if β is a product of a loop β' and a commutator of two other loops, then β and β' act on K in the same way. This shows that the isotopy type of K' depends only on K and the homology class $[\beta] \in H_1(F)$. When we deform K inside a small ball in N, we may shift β out of the projection of this ball in F so that the action of β does not interfere with the deformation of K. This implies that the isotopy class of K' depends only on the isotopy class of K and the homology class of β.

To sum up, we have constructed an action of $H_1(F)$ on the set of isotopy types of links in N so that links belonging to one $H_1(F)$-orbit always produce the same shlinks on F.

Theroem 3.2. *Let F be an oriented closed surface and let $p: N \to F$ be an oriented circle bundle over F. Then the mapping which associates with each link $K \subset N$ its shlink $S(K)$ on F establishes a bijective correspondence between the set of isotopy types of links in N modulo the action of $H_1(F)$ and the set of integral shlinks on F with total gleam $-\chi(p)$.*

Theorem 3.2 will be proven in subsection e. Theorem 3.2 gives a 2-dimensional reduction of the theory of links in N, considered up to isotopy and the action of $H_1(F)$.

Since the group $H_1(S^2)$ is trivial we get the following corollary.

Corollary 3.3. *Let $p: N \to S^2$ be an oriented circle bundle over the oriented 2-sphere. Then the mapping $K \mapsto S(K)$ establishes a bijective correspondence between the set of isotopy types of links in N and the set of integral shlinks on S^2 with total gleam $-\chi(p)$.*

Recall that the total spaces of circle bundles over S^2 are $S^2 \times S^1$, S^3, $\mathbb{R}P^3$, and the lens spaces $L(n, 1)$, $n = 3, 4, \cdots$. The corresponding Euler numbers are respectively 0, ± 1, ± 2, and $\pm n$, where the indeterminacy in sign is due to two possible orientations of the circle bundles. Corollary 3.3 completely reduces knot theory for these manifolds to a study of shlinks on S^2.

d. Proof of Theorem 2.1 modulo Theorem 3.2. We will deduce Theorem 2.1 from Theorem 3.2. Let $p: N \to S^2$ be the (locally trivial) oriented circle bundle over S^2 with the Euler number n. Such a bundle is known to exist for each $n \in \mathbb{Z}$ and to be unique up to bundle isomorphism. Note that $\pi_1(N) = \mathbb{Z}/|n|\mathbb{Z}$ and that the $|n|$-sheeted (universal) covering space of N is S^3.

Let $B \times [0, 1]$ be a normal cylinder in N with respect to p (see subsection b). Clearly, the interior of this cylinder is homeomorphic to $\mathbb{R}^3$. Denote by f the composition of an orientation preserving homeomorphism $\mathbb{R}^3 \to \mathrm{Int}(B \times [0, 1])$ and the inclusion of $\mathrm{Int}(B \times [0, 1])$ into N.

Each link K in $\mathbb{R}^3$ gives rise to the link $f(K)$ in N. This produces two integral shlinks on S^2: the shlink $S(K)_n$ (see §2.b) and the shlink $S(f(K))$ defined in subsection a via the bundle mapping $p: N \to S^2$. Let us show that

$$S(K)_n = S(f(K)). \tag{3.1}$$

Let $\mathscr{D}$ be a diagram of K on $\mathbb{R}^2$. Its underlying family of loops divides $S^2 = \mathbb{R}^2 \cup \{\infty\}$ into several "bounded" regions and one "unbounded" region containing $\{\infty\}$. The shlink $S(K)_n$ is represented by the shadow $s(\mathscr{D})_n$ on S^2, where the symbol n means that the gleam of the unbounded region has been increased by n so that the total gleam of $s(\mathscr{D})_n$ equals n. Put the diagram $\mathscr{D}$ inside the disc B. This diagram specifies a link $L \subset B \times [0, 1] \subset N$ which is clearly isotopic to $f(K)$. The shadow $s(L)$ of L (in the sense of subsection a) has the same underlying family of loops as $s(\mathscr{D})$, the same gleams of bounded regions (cf. subsection a), and the same total gleam as $s(\mathscr{D})_n$. Therefore, $s(L) = s(\mathscr{D})_n$. This yields (3.1).

Let us prove injectivity of the mapping $K \mapsto S(K)_n$. If $S(K)_n = S(K')_n$, then equality (3.1) and Corollary 3.3 imply that the links $f(K)$ and $f(K')$ are isotopic in N. Lifting an isotopy between these links to the universal covering of N we get an isotopy between K and K' in S^3. If $n = \pm 1$, then $N = S^3$ and each link in N is isotopic to $f(K)$ for a certain link K in $\mathbb{R}^3$. Therefore the surjectivity in Corollary 3.3 gives the surjectivity in Theorem 2.1.

e. Proof of Theorem 3.2. Let us first show that each integral shlink S on F with total gleam $-\chi(p)$ corresponds to a certain link in N. This will imply surjectivity in the statement of the theorem. Let s be a shadow on F representing S. Applying the move $(S2)^{-1}$ to s we may guarantee that all regions of S are discs. Thus, s gives rise to a CW-decomposition of F whose 0-cells are crossing points of s, 1-cells are edges of s (§1.a), and 2-cells are regions of s. Orient the 1-cells of this cell-decomposition in an arbitrary way.

We shall construct a generic link in N whose shadow on F equals s. Clearly there exists a generic link $K \subset N$ which projects onto the underlying family of loops of s so that the shadows s and $s(K)$ have the same underlying families of curves and the same regions. For any region X of s we define δ_X to be the difference between the gleams of X with respect to s and $s(K)$. The sum of the integers δ_X over all regions X of s is equal to $\chi(p) - \chi(p) = 0$. The formula $X \mapsto \delta_X$ defines an integral 2-cocycle of our cell-decomposition which presents zero cohomology class in $H^2(F; \mathbb{Z}) = \mathbb{Z}$. Therefore this cocycle is the coboundary of a certain 1-cochain $d \mapsto \nu_d \in \mathbb{Z}$, where d runs over the (oriented) 1-cells of our cell-decomposition of F. Changing the orientations of 1-cells, if necessary, we may assume that $\nu_d \geq 0$ for each 1-cell d. Let us change K over each 1-cell d by applying the fiber-fusion ν_d times to the segment of K which lies over d. This gives a new generic link in N whose shadow is easily seen to coincide with s.

Let us show now that if two generic links K, K' in N produce the same shadow on F, then the isotopy classes of K, K' are transformed into each other by the action of $H_1(F)$. Since $p(K) = p(K')$, we may vertically deform K' so that the parts of K, K' which lie over a small neighborhood of the set of crossing points of $p(K)$ are the same. Let d be an edge of $p(K)$. Since K and K' coincide over the ends of d, one may obtain the arc of K' lying over d from the arc of K lying over d via several fiber-fusions corresponding to a certain orientation of d and fulfilled over certain "blowing-up" points of d. If X is a region of the shadow $s(K) = s(K')$, then the gleams of X with respect to $s(K)$ and $s(K')$ are the same. Therefore, when we run along the edges of $p(K)$ which bound X, the total number of these fiber-fusions (with the signs determined by the orientations) must be equal to zero. Thus we may always connect the blowing-up points of these edges by oriented arcs inside X. The union of these arcs over all regions X gives us a system of oriented loops on F. The homology class of this system of loops transforms the isotopy type of K into that of K'.

To finish the proof we have to show that any two generic links in N with isotopic shadows on F are isotopic in N up to the action of $H_1(F)$. Observe first that if a shadow s is obtained from the shadow $s(L)$ of a link $L \subset N$ by a single move Si or $(Si)^{-1}$, $I = 1, 2, 3$, then there exists a link $L' \subset N$ isotopic to L such that $s = s(L')$. Indeed, over the part of $s(L)$ which is being changed by the move, L may be vertically deformed to lie in a normal cylinder so that the corresponding Reidemeister move Ωi or $(\Omega i)^{-1}$ is applicable to L inside this cylinder. This latter move produces the link L' with $s(L') = s$. (In the case of $(S2)^{-1}$ some additional fiber-fusions may be necessary to get the right L'.)

If the shadows of two generic links K, $K' \subset N$ are isotopic, then the preceding observation shows that K is isotopic to a link, say, K'' with $s(K'') = s(K')$. Thus, the isotopy classes of K' and K'' lie on the same orbit of the $H_1(F)$-action. This implies injectivity in the statement of Theorem 3.2.

f. Remarks. 1. The proof of Theorem 3.2 shows that each integral shadow on F with total gleam $-\chi(p)$ is the shadow of a certain generic link in N.

2. It would be of interest to generalize the technique of the paper to the case of links in Seifert fibered 3-manifolds and shadows on 2-dimensional orbifolds.

3. It is easy to transfer the results of subsection a and Theorem 3.2 to the case of compact F with $\partial F \neq \emptyset$. (Note that in this case the bundle $p: N \to F$ is automatically isomorphic to the projection $F \times S^1 \to F$.) One should assume from the very beginning that there is some fixed section of p over ∂F. Similar ideas work for noncompact F though the role of ∂F should be played by the union of ends of F.

4. Framed links and framed shlinks

a. Framed links. The notion of framed link is a well-known elaborated version of the notion of link. A framed link in a 3-manifold N is a link $K \subset N$ equipped with a nonsingular normal vector field on K considered up to homotopy in the class of nonsingular normal vector fields on K. The notion of isotopy readily extends to framed links. Note that sublinks of framed links are also framed.

It is easy to visualize framed links in the Euclidean 3-space $\mathbb{R}^3$ or in the 3-sphere $S^2 = \mathbb{R}^3 \cup \{\infty\}$. Indeed, each link diagram on $\mathbb{R}^2$ actually presents a framed link: one provides the diagram with the unit vector field which looks upwards. It is well known that each framed link in S^3 is isotopic to a framed link presented by some diagram on $\mathbb{R}^3$. (Just represent the link by any diagram, and then insert in the diagram a certain number of local twists, shown in Figure 10 (next page), to gain the correct framing.) It is also well known that two link diagrams on $\mathbb{R}^2$ present isotopic framed links if and only if they may be obtained from each other

FIGURE 10

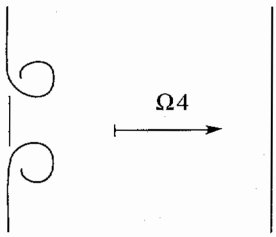

FIGURE 11

by the moves $\Omega 2$, $\Omega 3$, $\Omega 4$ (see Figures 6 and 11) and their inverses. Similar assertions hold true for framed links in the cylinder $F \times [0, 1]$ with the only difference that instead of diagrams on $\mathbb{R}^2$ one must speak of diagrams on F.

b. Framed shlinks. We define two local moves on shadows (see Figure 12). Two shadows (over an abelian group $A \supset \mathbb{Z}$) on the surface F are called regularly isotopic if they may be transformed into each other by a sequence of moves $S2$, $S3$, $S4$ and their inverses. We claim that regularly isotopic shadows are isotopic. This follows from the fact that $S4$ is a composition of $S1$ and $S1_{-}$, and $S1_{-}$ is a composition of $(S1)^{-1}$, $(S2)^{-1}$, $S3$, and $S2$ (see Figure 3).

Regular isotopy classes of shadows on F (over the group A) will be called framed shlinks on F over A. The claim above implies that there is the framing forgetting operation which associates the underlying shlink with each framed shlink. In particular, this enables us to speak of the total gleam of the framed shlink.

The definitions and results of §1 are easily transferred to the case of framed shlinks. In particular, one may speak of integral, rational, real, complex, and quaternionic framed shlinks. Subshlinks of framed shlinks

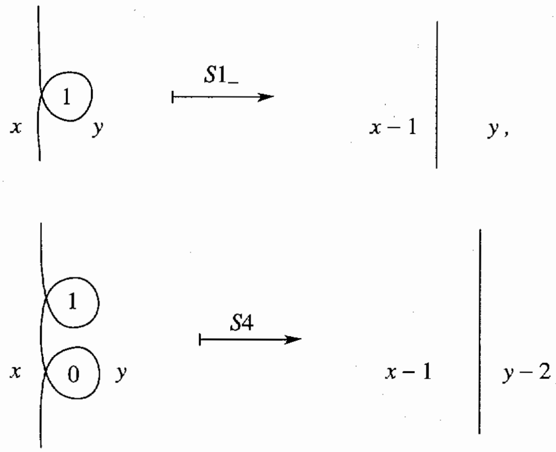

FIGURE 12

are also framed shlinks. Each framed shlink K on $\mathbb{R}^2$ with total gleam k extends to a framed shlink K_a on F with the gleam $k+a$ (for any $a \in A$).

Note that if two link diagrams $\mathscr{D}$, $\mathscr{D}'$ on F are related by the move $\Omega 4$, then their shadows $s(\mathscr{D})$, $s(\mathscr{D}')$ are related by the move $S4$. This implies that any link diagrams on F, presenting isotopic framed links in $f \times \mathbb{R}$, have regularly isotopic shadows. In this way each framed link K in $F \times \mathbb{R}$ determines a framed shlink $S(K)$ on F. Of course, the passage from K to $S(K)$ commutes with both the framing forgetting operation and the passage to sublinks. The total gleam of $S(K)$ equals zero.

Theorem 2.1 and Corollaries 2.2, 2.3 may be directly transferred to the present setting with the only difference that instead of links and shlinks one should speak of framed links and framed shlinks. The proof of Theorem 2.1 also works in the framed setting with the obvious changes. For completeness we state here the analogue of Corollary 2.2 for framed links.

Theorem 4.1. *The formula* $K \mapsto S(K)_1$ *establishes a bijective correspondence between the set of isotopy types of framed links in* $\mathbb{R}^3$ *and the set of integral framed shlinks on* S^2 *with total gleam* 1.

c. Framed links in S^1-bundles and their shlinks. Let F be an oriented closed surface and let $p: N \to F$ be an oriented circle bundle over F. Each framed link $K \subset N$ induces a framed shlink $S(K)$ on F as follows. It is easy to see that K is isotopic to a framed generic link $\overline{K}$ such that the framing of $\overline{K}$ is presented by the unit vector field tangent to the fibers of p and cooriented with the fibers. (One first deforms K into some framed generic link and then inserts local twists, as in Figure 10, to fulfill the condition on the framing.) The framed shlink $S(K)$ is defined to be the regular isotopy type of the shadow $s(\overline{K})$. Independence of $S(K)$ on the choice of $\overline{K}$ is proved in the same way as Lemma 3.1 though instead of $\Omega 1$ and $S1$ one must use $\Omega 4$ and $S4$. The transformation $K \mapsto S(K)$ commutes with both the forgetting of framing and the passage to sublinks.

In the same fashion as in §3 we define the action of $H_1(F)$ on the set of isotopy types of framed links in N. The only difference is that the role of generic links is played by the framed generic links with the framing tangent to the fibers of p and cooriented with the fibers. Note that forgetting of the framing is $H_1(F)$-equivariant.

The following are the analogues of Theorem 3.2 and Corollary 3.3.

Theorem 4.2. *The formula $K \mapsto S(K)$ establishes a bijective correspondence between the set of isotopy types of framed links in N modulo the action of $H_1(F)$ and the set of framed integral shlinks on F with total gleam $-\chi(p)$.*

The proof of this theorem is similar to the proof of Theorem 3.2.

Corollary 4.3. *Let $p: N \to S^2$ be an oriented circle bundle over the oriented 2-sphere. Then the formula $K \mapsto S(K)$ establishes a bijective correspondence between isotopy types of framed links in N and framed integral shlinks on S^2 with total gleam $-\chi(p)$.*

5. IRF-models and isotopy invariants of colored shadows

a. Colored links and shlinks. Fix a set I, the set of colors. An I-coloring (or, briefly, a coloring) of a link K in a 3-manifold is a function which associates with each component of K an element of I, called the color of the component. By an isotopy of colored links we will mean isotopy which preserves the colors of all components. Similar definitions are applied to framed links.

A coloring of a shadow s is a function which associates with each underlying loop of s an element of I, the color of the loop. By a (regular) isotopy of colored shadows we will mean a (regular) isotopy which preserves the colors of all underlying loops. A colored shlink is an isotopy class of colored shadows. Similarly, a colored framed shlink is a regular isotopy type of colored shadows.

If K is a colored (framed) link in the total space of an oriented circle bundle over the surface F, then the coloring of K in the obvious way descends to $S(K)$ so that $S(K)$ becomes a colored (framed) shlink on F.

b. The initial data. In this subsection we introduce some "initial data" which will be used to define IRF–models for colored complex shlinks. Essentially, our definition of the initial data is an axiomatization of certain properties of quantum $6j$-symbols (see [6]).

Let I be a finite set. Let u, v be two functions $I \to \mathbb{C}$ which associate with each $i \in I$ complex numbers u_i, v_i with $v_i \neq 0$ for all i. Assume that we have fixed a certain set of triples $i, j, k \in I$, called admissible triples, and also that any permutation of an admissible triple produces again an admissible triple.

An ordered 6-tuple $(i, j, k, l, m, n) \in I^6$ will be called admissible if the triples (i, j, k), (k, l, m), (m, n, i), and (j, l, n) are admissible. We assume that with each admissible 6-tuple (i, j, k, l, m, n) one associates a complex number, called the symbol of this tuple and denoted by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}.$$

The symbol is supposed to satisfy the following symmetry identity:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} l & n & j \\ i & k & m \end{vmatrix}. \tag{5.1}$$

This completes the definition of initial data.

We will say that the initial data satisfy condition $(*)$ if for any $j_1, j_2, j_3, j_4, j_5 \in I$ with admissible triples (j_1, j_3, j_4) and (j_2, j_4, j_5) we have

$$v_{j_4} \sum_{j \in I} v_j \begin{vmatrix} j_2 & j_3 & j \\ j_1 & j_5 & j_4 \end{vmatrix} \cdot \begin{vmatrix} j_1 & j_3 & j_4 \\ j_2 & j_5 & j \end{vmatrix} = 1, \tag{5.2}$$

and, for any $j_1, j_2, \cdots, j_6 \in I$ with $j_6 \neq j_4$,

$$\sum_{j \in I} v_j \begin{vmatrix} j_2 & j_3 & j \\ j_1 & j_5 & j_4 \end{vmatrix} \cdot \begin{vmatrix} j_1 & j_3 & j_6 \\ j_2 & j_5 & j \end{vmatrix} = 0. \tag{5.3}$$

Here and below it is understood that we sum up only those expressions which are defined, i.e, which include the symbols of admissible 6-tuples only.

We will say that the initial data satisfy condition $(**)$ if for any $j_1, j_2, j_3, a, b, c, d, e \in I$ we have

(5.4)

$$\sum_{j\in I} v_j \exp(u_b + u_d + u_f + u_j) \begin{vmatrix} j_2 & a & j \\ j_1 & c & b \end{vmatrix} \cdot \begin{vmatrix} j_3 & j & e \\ j_1 & d & c \end{vmatrix} \cdot \begin{vmatrix} j_3 & a & f \\ j_2 & e & j \end{vmatrix}$$

$$= \sum_{j\in I} v_j \exp(u_a + u_c + u_e + u_j) \begin{vmatrix} j_3 & b & j \\ j_2 & d & c \end{vmatrix} \cdot \begin{vmatrix} j_3 & a & f \\ j_1 & j & b \end{vmatrix} \cdot \begin{vmatrix} j_2 & f & e \\ j_1 & d & j \end{vmatrix}.$$

We will say that the initial data satisfy condition $(***)$ if for each $i \in I$ there exists a nonzero complex number Q_i such that for any admissible triple (i, a, b)

$$\sum_{j\in I} v_j \exp(-u_a - u_j + 2u_b) \cdot \begin{vmatrix} i & j & b \\ i & a & b \end{vmatrix} = Q_i, \tag{5.5}$$

$$\sum_{j\in I} v_j \exp(u_a + u_j - 2u_b) \cdot \begin{vmatrix} i & j & b \\ i & a & b \end{vmatrix} = Q_i^{-1}. \tag{5.6}$$

An example of initial data satisfying these conditions will be given in §6.

c. The state model. Fix the initial data $I, u, v \cdots$ as described in subsection b. Let s be an I-colored complex shadow on the oriented surface F. By an area-coloring of s we will mean an arbitrary mapping from the set of regions of s into the set I. An area-coloring of s is called admissible if for any edge d of s the colors of two regions of s, adjacent to d, and the (fixed) color of the loop of s, containing d, make an admissible triple. Denote the set of admissible area-colorings of s by $\mathrm{ad}(s)$.

Let $e_1, \cdots, e_q$ be the crossing points of s. Each point e_r is an intersection point of two (possibly coinciding) loops of s with the colors, say, i and l. Each admissible area-coloring $\eta \in \mathrm{ad}(s)$ provides us with the colors j, k, m, n of the four regions of s incident to e_r (see Figure 13). Put

$$|e_r^\eta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in \mathbb{C}.$$

Note that admissibility of η ensures admissibility of the 6-tuple (i, j, k, l, m, n). The equalities (5.1) imply that $|e_r^\eta|$ is correctly defined.

Let $X_1, \cdots, X_\mu$ be the regions of s. Let x_t, χ_t, and z_t be respectively the gleam, the Euler characteristic, and the number of corners (or edges) of the region X_t. Let $x_t' = x_t - z_t/2$ be the modified gleam of X_t (cf. Remark e.2 of §1). For each $\eta \in \mathrm{ad}(s)$ set

$$|s|_\eta = \prod_{r=1}^{q} |e_r^\eta| \times \prod_{t=1}^{\mu} ((v_{\eta(X_t)})^{\chi_t} \exp(2u_{\eta(X_t)} x_t')). \tag{5.7}$$

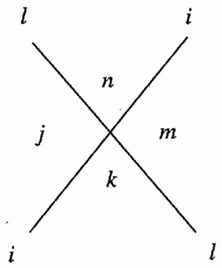

FIGURE 13

Finally, our state sum on s is

(5.8) $$|s| = \sum_{\eta \in \mathrm{ad}(s)} |s|_\eta \in \mathbb{C}.$$

There is an equivalent but sometimes more convenient expression for $|s|_\eta$. Namely, for each $i \in I$ denote by $\chi(\eta, i)$ the Euler characteristic of the union of the (open) regions of s whose color under the coloring η equals i. Denote by $x'(\eta, i)$ the sum of modified gleams of these regions. It is obvious that

$$|s|_\eta = \prod_{r=1}^{q} |e_r^\eta| \times \prod_{i \in I} (v_i^{\chi(\eta, i)} \exp(2u_i x'(\eta, i))).$$

Theorem 5.1. *Let s be an I-colored complex shadow on the oriented surface F. If the initial data satisfy conditions $(*)$ and $(**)$, then $|s|$ is invariant under the color-preserving moves $S2$ and $S3$. If the initial data satisfy the condition $(***)$, then for any colored shadow s_+ (resp. s_-), obtained from s by a single application of $(S1)^{-1}$ (resp. $(S1_-)^{-1}$) to a component of s with the color i, we have*

$$|s_+| = Q_i|s| \quad (\textit{resp. } |s_-| = Q_i^{-1}|s|).$$

Corollary 5.2. *If the initial data satisfy condition $(***)$, then $|s|$ is invariant under the color-preserving move $S4$.*

Theorem 5.1 and its corollary show that each initial data which satisfy conditions $(*)$, $(**)$, and $(***)$ give rise to a regular isotopy invariant of colored complex shadows on any oriented surface F. In other words we have a $\mathbb{C}$-valued invariant of colored framed complex shlinks on F. This produces via the constructions of §§2–4 a $\mathbb{C}$-valued isotopy invariant of colored framed links in $F \times \mathbb{R}$ and in S^1-bundles over F.

Note that, as follows from Theorem 5.1, a single twist of the framing of a link along a component of color i in the positive (resp. negative) direction leads to multiplication of the invariant by Q_i (resp. by Q_i^{-1}).

d. Proof of Theorem 5.1. First we redraw the moves $S1$, $S1_-$, $S2$, $S3$ with the modified gleams (see Figure 14 where α, β, γ, δ, ε, μ are the modified gleams, and j_1, j_2, j_3, i are the colors of loops).

Let us prove the invariance of $|s|$ under the move $S2$. Let the shadow s' be obtained from the shadow s by a single application of $S2$. Let X, X_1, X_2 be the regions of s, marked respectively by $0, \beta, \delta$ in Figure 14.

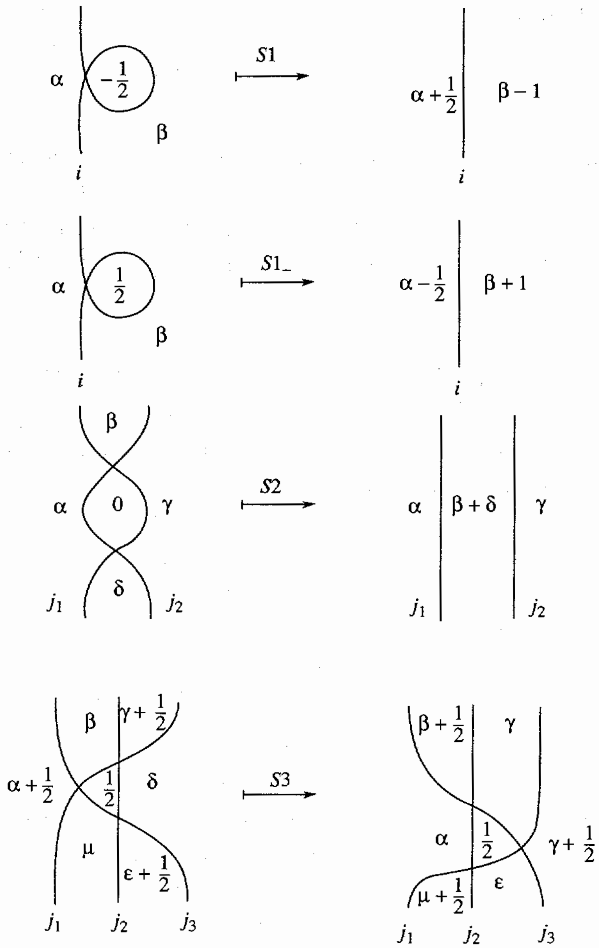

FIGURE 14

These regions give rise to a region Y of s', marked by $\beta+\delta$ in Figure 14. Clearly,

$$\chi(X)=\chi(X_1\cup X_2)-\chi(Y)=1,$$

which corresponds to the appearance of v_j and v_{j_4} in condition $(*)$. Fix the colors of all regions of s except X, and vary the color j of X. The convenient notation for the colors of the regions of s involved in Figure 14 is the following: the colors of regions marked by α, β, γ, δ are respectively j_3, j_4, j_5, j_6. Equality (5.3) ensures that if the colors j_4, j_6 of the regions X_1, X_2 are distinct, then the sum of the expressions $|s|_\eta$ over all $j\in I$ is equal to zero. If $j_4=j_6$, then equality (5.2) ensures that the same sum equals $|s'|_\eta$, where η is the area-coloring of s' induced by the coloring of the regions of s distinct from X. Summing up all these equalities we get $|s|=|s'|$.

The other claims of Theorem 5.1 are proved along the same lines. The convenient notation for the colors of regions which establishes the exact correspondence with the conditions $(**)$ and $(***)$ is the following: the colors of regions marked by α, β, γ, δ, ε, μ, and $\mp\frac{1}{2}$ should be respectively denoted by a, b, c, d, e, f, and j.

6. Quantum invariants of colored shlinks

a. The initial data associated with $U_q(\mathrm{sl}_2)$. Representation theory for the Hopf algebra $U_q(\mathrm{sl}_2(\mathbb{C}))$ naturally leads to the so-called quantum $6j$-symbols (or $(q-6j)$-symbols) which essentially satisfy our conditions $(*)$, $(**)$, and $(***)$. We describe the initial data obtained in this way, referring to [6] for details on $(q-6j)$-symbols.

Fix an integer $r\geq 3$ and denote by I the set $\{0, 1/2, 1, \cdots, (r-3)/2, (r-2)/2\}$. Fix a primitive complex root Q of 1 of degree $4r$, so that $Q=\exp(\pi\sqrt{-1}h/2r)$ with $h\in\mathbb{Z}$. For each integer $n\geq 1$ put

$$[n]=\frac{Q^{2n}-Q^{-2n}}{Q^2-Q^{-2}}\in\mathbb{R}$$

and

$$[n]!=[n][n-1]\cdots[2][1].$$

In particular, $[1]!=[1]=1$. Put also $[0]!=[0]=1$. Note that $[n]!=0$ for $n\geq r$ and $[n]!\neq 0$ for $n<r$.

A triple $(i, j, k)\in I^3$ is admissible if $i+j+k$ is an integer, $i+j+k\leq r-2$, and $i\leq j+k$, $j\leq i+k$, $k\leq i+j$. For each admissible triple

i, j, k set

$$\Delta(ijk) = \left(\frac{[i+j-k]![i-j+k]![-i+j+k]!}{[i+j+k+1]!}\right)^{1/2}.$$

Here by the square root of a real number a we mean $a^{1/2} \geq 0$ if $a \geq 0$ and $\sqrt{-1}|a|^{1/2}$ if $a < 0$.

Recall the notion of admissible 6-tuple (see §5.b). For any admissible 6-tuple $(i, j, k, l, m, n) \in I^6$ one defines the Racah-Wigner $6j$-symbol which is computed as follows:

$$\begin{aligned}
&\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{\mathrm{RW}} \\
&\quad = \Delta(ijk)\Delta(imn)\Delta(ljn)\Delta(lmk) \\
&\quad\cdot \sum_z (-1)^z [z+1]! \{[z-i-j-k]![z-i-m-n]! \\
&\quad\cdot [z-l-j-n]![z-l-m-k]![i+j+l+m-z]! \\
&\quad\cdot [i+k+l+n-z]![j+k+m+n-z]!\}^{-1}.
\end{aligned}$$

Here z runs over nonnegative integers such that all expressions in the square brackets are nonnegative.

Set

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{\mathrm{RW}}.$$

This number is either real or purely imaginary.

We have the following obvious symmetry relations:

$$(6.1)\qquad \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix},$$

which imply (5.1).

For $j \in I$ we put

$$u_j = \pi\sqrt{-1}\, j(1 - h(j+1)r^{-1}), \qquad v_j = (-1)^{2j}[2j+1].$$

(Recall that $Q = \exp(\pi\sqrt{-1}h/2r)$.) This completes the description of the initial data.

Theorem 6.1. *The initial data* $(I, u, v, \cdots)$ *described above satisfy conditions* $(*)$ $(**)$, *and* $(***)$ *with* $Q_i = (-1)^{2i} Q^{4i(i+1)}$ *for* $i \in I$.

The state sum (5.8) defined via the initial data described above will be denoted by $J_s(Q)$. Theorems 5.1 and 6.1 imply that the complex number $J_s(Q)$ is preserved under regular isotopies of s and gives in this way a $\mathbb{C}$-valued invariant of framed shlinks. If S is the framed colored shlink presented by a colored shadow s, then $J_s(Q)$ does not depend on the choice of s, and so we may define

$$J_S(Q) = J_s(Q).$$

b. Proof of Theorem 6.1. Theorem 6.1 basically follows from equalities 6.16, 6.19, 6.20 in [6]. These equalities are formulated in terms of $(q-6j)$-symbols which are related to the Racah-Wigner $6j$-symbols by the formula

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix} = [2k+1]^{1/2}[2n+1]^{1/2}(-1)^{l+m+2k-i-j} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{\mathrm{RW}},$$

which implies that

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix} = \sqrt{-1}^{\,2k+2n}[2k+1]^{1/2}[2n+1]^{1/2} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}. \tag{6.2}$$

Substituting this in 6.16, 6.19, and 6.20 of [6] one gets the desired conditions $(*)$, $(**)$, and $(***)$. However, one should be cautious since [6] contains inaccuracies. There are in particular wrong signs in 6.19. The correction leads to the following version of 6.19:

$$\begin{aligned} &\sum_{g\in I}(-1)^{-a+b+g+f} q^{(c_a-c_b-c_g-c_f)/2} \\ &\quad\cdot \begin{Bmatrix} j_2 & a & g \\ j_1 & c & b \end{Bmatrix} \begin{Bmatrix} j_3 & g & e \\ j_1 & d & c \end{Bmatrix} \begin{Bmatrix} j_3 & a & f \\ j_2 & e & g \end{Bmatrix} \\ &= \sum_{g\in I}(-1)^{-d+c+g+e} q^{(c_d-c_c-c_g-c_e)/2} \\ &\quad\cdot \begin{Bmatrix} j_3 & b & g \\ j_2 & d & c \end{Bmatrix} \begin{Bmatrix} j_3 & a & f \\ j_1 & g & b \end{Bmatrix} \begin{Bmatrix} j_2 & f & e \\ j_1 & d & g \end{Bmatrix}, \end{aligned}$$

where $q = Q^4$ and $c_i = i(i+1)$. Substituting

$$(-1)^i q^{-c_i/2} = \exp(u_i) \tag{6.3}$$

and (6.2), and replacing g by j, we get (5.4). There are also wrong signs in the formulas 6.10, 6.11, and 6.14 of [6], the correct signs being respectively

$$(-1)^{j_1-j_2-j_{12}}, \quad (-1)^{j_1+j_2-j_{12}}, \quad (-1)^{j_1-j_2-j_{12}}.$$

This leads to the following correct version of 6.20: for any $\varepsilon = \pm 1$ and any admissible triple $(a, b, i) \in I^3$

$$\sum_{j\in I}(-1)^{\varepsilon(j-a)} q^{\varepsilon(c_a+c_j-2c_b)/2} \frac{[2j+1]}{[2b+1]} \begin{Bmatrix} i & j & b \\ i & a & b \end{Bmatrix}$$
$$= ((-1)^{2i} q^{c_i}))^{\varepsilon} = ((-1)^{2i} Q^{4i(i+1)})^{\varepsilon} = Q_i^{\varepsilon}.$$

Substituting (6.2) and (6.3), we get (5.5) and (5.6).

Note also that the paper [6] is concerned mostly with unrestricted IRF-models corresponding to generic q; for parallel results on the restricted models corresponding to roots of 1 see the last section of [6].

c. Examples. 1. Let γ be a simple (imbedded) closed curve on the oriented surface F such that the complement of γ in F is connected. Provide γ with a color $j \in I$ and provide the complement of γ with a gleam $x \in \mathbb{C}$. This gives us a colored shadow s on F. If j is a half-integer but not an integer, then s has no admissible area-colorings and therefore $|s| = 0$. If $j \in \mathbb{Z} \cap I$, then one easily computes that

$$|s| = \sum_{\substack{i\in I \\ r-2-j\geq 2i\geq j}} v_i^{\chi(F\setminus\gamma)} \exp(2xu_i),$$

where χ is the Euler characteristic.

2. Let γ be a simple closed curve in F which splits F into two surfaces F_1 and F_2. Equip γ with the color $1/2$, and equip the regions $\operatorname{Int} F_1$, $\operatorname{Int} F_2$ with the gleams x, $y \in \mathbb{C}$. This gives us a colored shadow s on F with the only underlying closed curve γ and the total gleam $x + y$. The admissible area-colorings of s associate certain $i \in I$ with $\operatorname{Int} F_1$, and either $i - 1/2$ or $i + 1/2$ with $\operatorname{Int} F_2$. Thus,

$$|s| = \sum_{\substack{i\in I \\ i\neq 0}} v_i^{\chi_1} v_{i-1/2}^{\chi_2} \exp(2xu_i + 2yu_{i-1/2})$$
$$+ \sum_{\substack{i\in I \\ i\neq (r-2)/2}} v_i^{\chi_1} v_{i+1/2}^{\chi_2} \exp(2xu_i + 2yu_{i+1/2}),$$

where $\chi_1 = \chi(F_1)$ and $\chi_2 = \chi(F_2)$.

Let us take $r = 3$ and $Q = \exp(\pi\sqrt{-1}/6)$. Then $I = \{0, 1/2\}$, $v_0 = 1$, $v_{1/2} = -[2] = -(Q^2 + Q^{-2}) = -1$, $u_0 = 0$, and $u_{1/2} = \pi\sqrt{-1}/4$. One easily computes

$$|s| = (-1)^{\chi_1} \exp(\pi\sqrt{-1}x/2) + (-1)^{\chi_2} \exp(\pi\sqrt{-1}y/2). \tag{6.4}$$

This complex number $|s|$ is invariant under regular isotopies. The same number considered up to multiplication by integral powers of $Q_{1/2} = -Q^3 = -\sqrt{-1}$ is invariant under arbitrary isotopies.

This example shows that on any oriented surface F there exist real shadows nonisotopic to rational shadows. Indeed, when we vary x and y in $\mathbb{R}$, the number $|s|$ runs over a noncountable set, whereas the set of rational shlinks is easily seen to be countable. Similarly, there exist complex shadows nonisotopic to real shadows (cf. also Remark f.1 below).

d. A recurrent formula for $J_s(Q)$. In this subsection we present a recurrent formula which may be used to compute $J_s(Q)$. This formula is applicable in the case when certain components of the shadow s are colored by $1/2 \in I$. In the case where all components are colored by $1/2$, the formula enables one to reduce the computation of $J_s(Q)$ to the case where s consists of simple disjoint loops on F. The results of this subsection are closely related to the Kauffman model for the Jones polynomial of links; this relationship will be discussed in subsection e.

Let us say that three I-colored complex shadows s, s_+, s_- on F form a splitting triple if there is a 2-disc $B \subset F$ such that s, s_+, s_- coincide outside B and look as in Figure 15 inside B, where the colors of the components of s, s_+, s_- which meet B are equal to $1/2 \in I$. Note that the symbols $x, y, z, t, y+t-2$, $y-1$, $t-1$, $x+z$ in Figure 15 represent the (nonmodified) gleams of the corresponding regions (under the convention exhibited in §1.a).

The local picture of a splitting triple with modified gleams (cf. Remark e.2 of §1) is represented in Figure 16 (next page). This picture reveals the skew-symmetry between s_+ and s_-.

Theorem 6.2. *If s, s_+, s_- is a splitting triple of I-colored complex shadows on the oriented surface, then*

$$J_s(Q) = QJ_{s_+}(Q) + Q^{-1}J_{s_-}(Q). \tag{6.5}$$

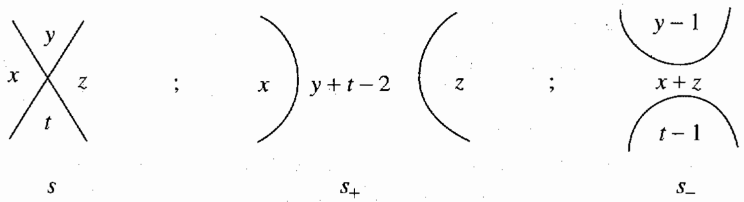

FIGURE 15

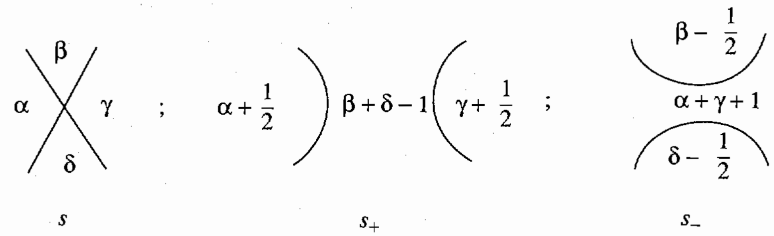

FIGURE 16

Proof. Let us first compute all symbols

(6.6) $$\begin{vmatrix} \frac{1}{2} & j & k \\ \frac{1}{2} & m & n \end{vmatrix}$$

with $j, k, m, n \in I$. Applying the symmetry relations (6.1), if necessary, we may assume that $j \leq m \leq n$ and $k \leq n$. Admissibility of the triple $1/2, m, n$ implies that $1/2 + m + n \in \mathbb{Z}$ and therefore that $m \neq n$. Hence $m \leq n - 1/2$. Admissibility of the triple $1/2, j, n$ implies that $j \geq n - 1/2$. Since $j \leq m$, we have $j = m = n - 1/2$. Also, admissibility of the triple $1/2, j, k$ implies that

$$n \geq k \geq j - 1/2 = n - 1$$

and $k + n \in \mathbb{Z}$. Therefore either $k = n$, or $k = n - 1$. Thus, up to symmetries (6.1), the symbol (6.6) equals

(6.7) $$\begin{vmatrix} \frac{1}{2} & n - \frac{1}{2} & n + \varepsilon \\ \frac{1}{2} & n - \frac{1}{2} & n \end{vmatrix},$$

where $\varepsilon = 0$ or $\varepsilon = -1$. A direct computation shows that

(6.8) $$\begin{vmatrix} \frac{1}{2} & n - \frac{1}{2} & n + \varepsilon \\ \frac{1}{2} & n - \frac{1}{2} & n \end{vmatrix} = \begin{cases} (-1)^{2n-1}[2n]^{-1} & \text{if } \varepsilon = -1, \\ (-1)^{2n}[2n]^{-1}[2n+1]^{-1} & \text{if } \varepsilon = 0. \end{cases}$$

(The sum in the definition of $\{\ \}^{\mathrm{RW}}$ is reduced in this case to one nonzero term corresponding to $z = 2n$.)

Let us prove (6.5).

Denote the regions of s marked by x, y, z, t by X, Y, Z, T respectively. Let η be an admissible area-coloring of s. There are six possibilities for the η-colors of X, Y, Z, T given in the following table, in which n is an element of I:

In the cases I, II the area-coloring η in the obvious way determines an area-coloring η_+ of s_+ and, as we will show,

(6.9) $$|s|_\eta = Q|s_+|_{\eta_+}.$$

	I	II	III	IV	V	VI
X	$n-1$	n	$n-\frac{1}{2}$	$n-\frac{1}{2}$	$n-\frac{1}{2}$	n
Y	$n-\frac{1}{2}$	$n-\frac{1}{2}$	n	$n-1$	n	$n-\frac{1}{2}$
Z	n	$n-1$	$n-\frac{1}{2}$	$n-\frac{1}{2}$	$n-\frac{1}{2}$	n
T	$n-\frac{1}{2}$	$n-\frac{1}{2}$	$n-1$	n	n	$n-\frac{1}{2}$

Similarly, in the cases III, IV the area-coloring η determines an area-coloring η_- of s_-, and we have

(6.10) $$|s|_\eta = Q^{-1}|s_-|_{\eta_-}.$$

In the cases V, VI the coloring η determines both an area-coloring η_+ of s_+ and an area-coloring η_- of s_-, and we have

(6.11) $$|s|_\eta = Q|s_+|_{\eta_+} + Q^{-1}|s_-|_{\eta_-}.$$

Summing up (6.9), (6.10), (6.11) over all admissible area-colorings η of s we get (6.5).

The equalities (6.9), (6.10), and (6.11) follow directly from the definitions and the following equalities:

$$\begin{vmatrix} \frac{1}{2} & n-\frac{1}{2} & n-1 \\ \frac{1}{2} & n-\frac{1}{2} & n \end{vmatrix} = Qv_{n-1/2}^{-1}\exp(u_{n-1}+u_n-2u_{n-1/2}),$$

$$\begin{vmatrix} \frac{1}{2} & n-\frac{1}{2} & n-1 \\ \frac{1}{2} & n-\frac{1}{2} & n \end{vmatrix} = Q^{-1}v_{n-1/2}^{-1}\exp(2u_{n-1/2}-u_n-u_{n-1}),$$

$$\begin{vmatrix} \frac{1}{2} & n-\frac{1}{2} & n \\ \frac{1}{2} & n-\frac{1}{2} & n \end{vmatrix} = (Qv_n^{-1}+Q^{-1}v_{n-1/2}^{-1})\exp(2u_{n-1/2}-2u_n),$$

$$\begin{vmatrix} \frac{1}{2} & n-\frac{1}{2} & n \\ \frac{1}{2} & n-\frac{1}{2} & n \end{vmatrix} = (Qv_{n-1/2}^{-1}+Q^{-1}v_n^{-1})\exp(2u_n-2u_{n-1/2}).$$

All these formulas follow directly from (6.8). Actually, the second and fourth formulas are obtained respectively from the first and third ones by complex conjugation. This completes the proof of the theorem.

e. Relation to the Jones polynomial. V. Jones [3] introduced for each oriented link $K \subset \mathbb{R}^3$ a polynomial $V_K(t) \in \mathbb{Z}[\sqrt{t}, \sqrt{t}^{-1}]$. Kauffman [5] introduced a version $L_K(t)$ of this polynomial which is defined for every framed link $K \subset \mathbb{R}^3$. If the link K is both oriented and framed, then $L_K(t) = \delta t^a V_K(t^4)$, where the integer a is determined by the framing and the linking coefficients of the components of K, and $\delta = -(t^2 + t^{-2})$.

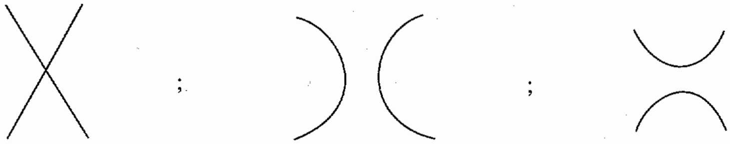

FIGURE 17

The polynomial L_K (and its value for each specified t) may be characterized by the following three properties: (i) if a diagram of a framed link $K' \subset \mathbb{R}^3$ is obtained from a diagram of a framed link $K \subset \mathbb{R}^3$ by inserting one positive (resp. negative) twist as in Figure 10, then

$$L_{K'}(t) = -t^3 L_K(t) \quad (\text{resp. } L_K(t) = -t^{-3} L_K(t));$$

(ii) if three framed links $K, K_+, K_- \subset S^3$ are represented by link diagrams in the plane which coincide outside some disc and look as in Figure 17 inside the disc, then

$$L_K(t) = tL_{K_+}(t) + t^{-1} L_{K_-}(t);$$

(iii) if K is a trivial knot in $\mathbb{R}^2$ with the framing orthogonal to $\mathbb{R}^2$, then $L_K(t) = \delta$.

Put

$$c_Q = \sum_{i \in I} [2i+1]^2 \exp(2u_i).$$

For example, if $r = 3$ and $Q = \exp(\pi\sqrt{-1}/6)$, then $c_Q = 1 + \sqrt{-1}$.

Theorem 6.3. *Let K be a framed link in $\mathbb{R}^3$, let $S = S(K)_1$ be the integral shlink on S^2 corresponding to K (see §2.b), and provide all components of S with the color $1/2 \in I$. Then*

$$J_S(Q) = c_Q L_K(Q),$$

where $J_S(Q)$ is the invariant of the colored shlink S defined in subsection a, *and $L_K(Q)$ is the value of $L_K(t)$ for $t = Q$.*

Theorem 6.3 shows that (whenever $c_Q \neq 0$) one may regard the set $\{J_S(Q)\}_Q$ as an extension of the Jones polynomial to complex shlinks.

In subsection g we will considerably generalize Theorem 6.3, extending it to colored framed links in $\mathbb{R}^3$.

Proof of Theorem 6.3. The mapping $K \mapsto J_S(Q)$ with $S = S(K)_1$ satisfies properties (i), (ii) of the polynomial L_K. This follows from the equality $Q_{1/2} = -Q^3$ (cf. the statement of Theorem 6.1) and the fact that Figure 15 is the shadow version of Figure 17. Thus $J_s(Q) = cL_K(Q)$ for a certain $c \in \mathbb{C}$. One computes c to be c_Q applying the latter equality to the trivial knot.

f. Remarks. 1. One may deduce from Theorems 6.1 and 6.3 that there exist rational shadows on S^2 with total gleam 1 nonisotopic to integral shadows. Indeed, these theorems imply that for each integral shadow s on S^2 with total gleam 1, the number $J_s(Q)$ must be a linear combination of $1, Q, Q^2, \cdots, Q^{4r-1}$ with integer coefficients. However, for $x, y = 1 - x \in \mathbb{Q}$ the expression (6.4), generally speaking, cannot be presented as such an integral linear combination of $1, Q, \cdots, Q^{11}$ with $Q = \exp(\pi\sqrt{-1}/6)$.

2. An interesting question regarding the invariants $\{J_s(Q)\}_Q$ of a colored shadow s on F is whether the mapping $Q \mapsto J(Q)$ extends continuously to the unit circle or not. Theorem 6.3 shows that for integral shadows on S^2 with total gleam 1 the answer is positive. One may show that the answer is positive for arbitrary integral shadows on S^2 with nonzero total gleam. Moreover, the mapping $Q \mapsto J_s(Q)$ is actually a polynomial on Q. The proof of these claims appeals to quite a different technique and will be presented elsewhere. The essential point is the existence (and uniqueness) of the Jones-type polynomial for framed links in lens spaces. For links in $\mathbb{R}P^3$ such a polynomial has been recently constructed by Ju. Drobotuchina and independently by J. Hoste and J. Przytycki.

g. Relation with invariants of colored links in $\mathbb{R}^3$. The Jones polynomial was generalized by several authors to an invariant of colored framed links in $\mathbb{R}^3$ (see for instance [9], [11] and references therein). The colors are irreducible representations of a ribbon quasitriangular Hopf algebra. In particular, with each primitive complex root Q of 1 of degree $4r$ one associates such a Hopf algebra over $\mathbb{C}$, denoted somewhat abussively by $U_q(\mathrm{sl}_2)$, where $q = Q^4$. There are canonical irreducible finite-dimensional representations of $U_q(\mathrm{sl}_2)$ indexed by the elements of the set $I = \{0, 1/2, 1, 3/2, \cdots, (r-2)/2\}$. Thus for each I-colored framed link $K \subset \mathbb{R}^3$ we have a $\mathbb{C}$-valued isotopy invariant of K denoted by $J_K(Q)$. (Actually one may show that the mapping $Q \mapsto J_K(Q)$ is given by a polynomial in Q, Q^{-1}.) In particular, when all the components of K are colored by $1/2 \in I$, then

$$J_K(Q) = L_K(Q),$$

where L_K is the Kauffman version of the Jones polynomial.

Theorem 6.4. *Let K be an I-colored framed link in $\mathbb{R}^3$, and let $S = S(K)_1$ be the corresponding shlink on S^2. Then*

$$J_S(Q) = c_Q J_K(Q).$$

This theorem generalizes Theorem 6.3. It also shows that our invariants of colored complex shlinks generalize the Jones-type invariants of I-colored links in $\mathbb{R}^3$.

Proof of Theorem 6.4. The proof is based on the computation of $J_K(Q)$ given in [6]. The computation goes as follows. Take a link diagram $\mathscr{D}$ in $\mathbb{R}^2$ which presents K. Deforming $\mathscr{D}$, if necessary, we may assume that $\mathscr{D}$ lies in general position with respect to the second (vertical) coordinate in $\mathbb{R}^2$. This means that this coordinate has a finite number of nondegenerate local minima and maxima on $\mathscr{D}$. We may also assume that in each crossing point of $\mathscr{D}$ the two branches of $\mathscr{D}$ are mutually orthogonal and make the angle 45° with the horizontal and vertical lines in $\mathbb{R}^2$. Let η be an admissible area-coloring of the shadow $s(\mathscr{D})$ on $\mathbb{R}^2$. With every local extremum of the second coordinate on $\mathscr{D}$ and with every crossing point of $\mathscr{D}$ we associate one of the following complex numbers, according to the rule exhibited in Figure 18:

(A) $\exp(u_{j_{13}} + u_{j_{12}} - u_j - u_{j_1})\overline{v}_{j_{13}}\overline{v}_{j_{12}} \begin{vmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{vmatrix}$,

(B) $\exp(-u_{j_{13}} - u_{j_{12}} + u_j + u_{j_1})\overline{v}_{j_{13}}\overline{v}_{j_{12}} \begin{vmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{vmatrix}$,

(C) $\sqrt{-1}^{2j_2}\overline{v}_{j_{12}}\overline{v}_{j_1}^{-1}$,

(D) $\sqrt{-1}^{-2j_2}\overline{v}_{j_{12}}\overline{v}_{j_1}^{-1}$,

where

$$\overline{v}_i = \sqrt{-1}^{2i}[2i+1]^{1/2},$$

j_2, j_3 are the (fixed) colors of the components of K, and j, j_1, j_{12}, j_{13} are the η-colors of the corresponding regions of $s(\mathscr{D})$. Clearly, $\overline{v}_i^2 = v_i$. Let $|\eta|$ be the product of all these complex numbers associated with local extrema and crossings of $\mathscr{D}$. Let U be the only unbounded region of $s(\mathscr{D})$ on $\mathbb{R}^2$. Then for any $i \in I$

$$J_K(Q) = \sum_{\substack{\eta \in \mathrm{ad}(S(\mathscr{D})) \\ \eta(U)=i}} |\eta|. \tag{6.12}$$

Three remarks are in order. First, formula (6.12) is stated in [6] only for the case $i = 0$. However, the arguments of [6] work for all $i \in I$. Second, the numbers (A), (B) associated here with the crossings have been associated in [6] with the mirror images of these crossings. This corrects

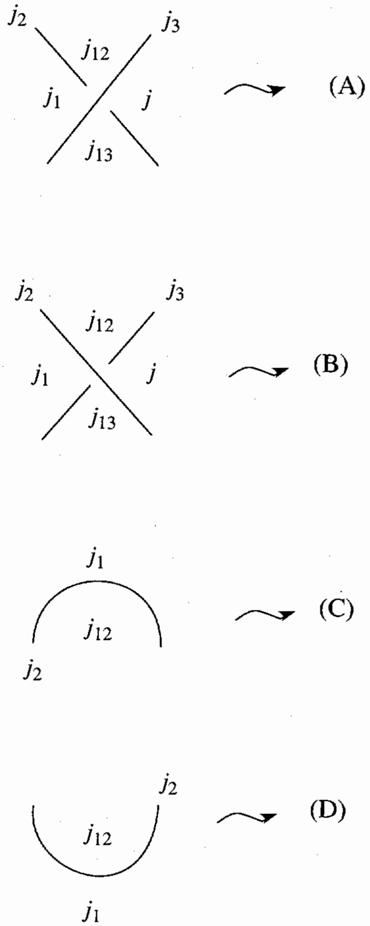

FIGURE 18

the uncommon definition of positive and negative crossings used in [6] to define $J_K(Q)$. We follow the standard conventions used, say, in [1]. Third, we have incorporated here the corrections to the formulas of [11] mentioned in subsection b.

The shadow $s(\mathscr{D})$ in $\mathbb{R}^2$ extends to the shadows $s(\mathscr{D})_0$ and $s(\mathscr{D})_1$ in S^2. The latter two shadows differ only in the gleam of the region $\overline{U} = U \cup \{\infty\}$ which is larger by 1 for $s(\mathscr{D})_1$. The admissible area-colorings of $s(\mathscr{D})$, $s(\mathscr{D})_0$, and $s(\mathscr{D})_1$ are the same. For each such coloring η with $\eta(\overline{U}) = i \in I$ we have

$$|s(\mathscr{D})_1|_\eta = \exp(2u_i)|s(\mathscr{D})_0|_\eta. \tag{6.13}$$

We shall prove that

$$(6.14) \qquad |s(\mathscr{D})_0|_\eta = v_i^2 |\eta|,$$

which together with (6.12) would imply the claim of the theorem:

$$J_S(Q) = |s(\mathscr{D})_1| = \sum_{i\in I} \sum_{\substack{\eta\in \mathrm{ad}(s(\mathscr{D}))\\ \eta(\overline{U})=i}} |s(\mathscr{D})_1|_\eta$$
$$= J_K(Q) \cdot \sum_{i\in I} v_i^2 \exp(2u_i) = J_K(Q) \sum_{i\in I} [2i+1]^2 \exp(2u_i).$$

To prove (6.14) we shall compare the definitions of $|\eta|$ and $|s(\mathscr{D})_0|_\eta$. Each crossing point e_r of $\mathscr{D}$ as in Figure 18 contributes the multiple

$$|e_r^\eta| = \begin{vmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{vmatrix}$$

to both products. It follows directly from the definition of the shadow of a diagram that the product of exponents $\exp(\dots)$, which are involved in the expressions (A), (B) associated with the crossing points, over all these points exactly equals the multiple

$$\prod_{t=1}^{\mu} \exp(2u_{\eta(X_t)} x_t')$$

which appears in (5.7).

Each closed curve in $\mathbb{R}^2$ obviously has the same number of local maxima and minima with respect to the second coordinate. Therefore the product of the expressions $\sqrt{-1}^{2j_2}$, $\sqrt{-1}^{-2j_2}$ which appear in (C), (D) equals 1.

It remains to compute the product of the multiples $\overline{v}_j$ which appear in (A), (B), (C), (D). Let X be a region of $s(\mathscr{D})_0$ and let l be the η-color of X. Each crossing point of $\mathscr{D}$ lying on $\partial \overline{X}$ contributes either $\overline{v}_l$, or $\overline{v}_l^{-1}$, or 1 to $|\eta|$ (see Figure 19, where X is shaded). An elementary application of the Morse theory implies that the product of these contributions equals

$$\overline{v}_l^{2\chi} = v_l^\chi,$$

where $\chi = \chi(X)$ is the Euler characteristic of X if X is a bounded region of $s(\mathscr{D})$, and $\chi = \chi(X) - 2$ if $X = U \cup \{\infty\}$. Since $i = \eta(\overline{U})$, the product of the multiples $\overline{v}_j$ differs from the corresponding term in $|s(\mathscr{D})_0|$ by v_i^{-2}. This implies (6.14) and completes the proof of Theorem 6.4.

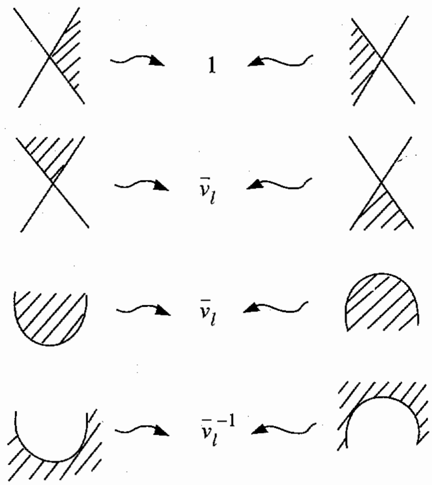

FIGURE 19

7. Shadow 3-manifolds and topological quantum field theory

It is well known that closed oriented 3-manifolds may be described in terms of framed links in S^3 and the Kirby-Fenn-Rourke moves on these links (see [1]). One may invert this procedure and define shadow 3-manifolds as the equivalence classes of framed shlinks modulo the shadow version of the Kirby-Fenn-Rourke moves. Moreover, one may define shadow 3-manifolds with boundary. The results of §6 show that the technique of [10] is applicable to shadow 3-manifolds. This technique produces a "shadow" topological quantum field theory in dimension 3. The author plans to explore this subject in more detail elsewhere.

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